Basics of Markov Chain Monte Carlo (MCMC)

Michael Anderson, PhD

Department of Biostatistics and Epidemiology
The University of Oklahoma Health Sciences Center

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Outline

1. Markov Chain (Andrei Markov 1907)
2. Gibbs Sampling (Geman and Geman 1984)
3. Metropolis Algorithm (Nicholas Metropolis 1953)
4. Monte Carlo Estimation (Stanislaw Ulam 1946)

Markov Chains

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- Most Markov chains we will consider will converge to a single stationary distribution as $n \to \infty$

Gibbs Sampling

Suppose we want to describe $p(\theta_1, \theta_2 | x_1, \ldots, x_n)$. Suppose further that we know $p(\theta_1 | \theta_2, x_1, \ldots, x_n)$ and $p(\theta_2 | \theta_1, x_1, \ldots, x_n)$.

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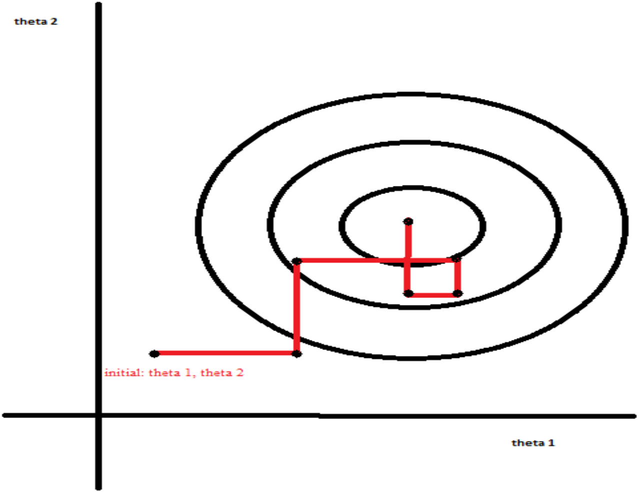
How it works:

1. Choose an initial value for θ_2 say $\theta_2^{(0)}$.
2. Obtain $\theta_1^{(1)}$ from $p(\theta_1 | \theta_2^{(0)}, x_1, \ldots, x_n)$.
3. Obtain $\theta_2^{(1)}$ from $p(\theta_2 | \theta_1^{(1)}, x_1, \ldots, x_n)$.
4. Repeat steps 2 and 3 with the new θs a large number of times.

Gibbs Sampling

This produces a Markov Chain that "explores" the parameter space.



Build Your Own Gibbs Sampler

F-35 Speed vs Accuracy

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$$X|Y = y \sim N(921 + 15^2 \frac{1}{3^2}(Y - 5), 100^2 - 15^2 \frac{1}{3^2} 15^2)$$

$$Y|X = x \sim N(5 + 15^2 \frac{1}{100^2}(X - 921), 3^2 - 15^2 \frac{1}{100^2} 15^2)$$

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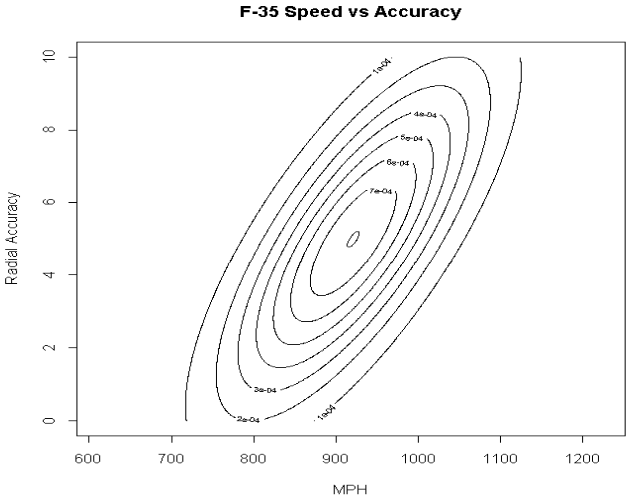

See the "F35 bivariate normal.R" file

Metropolis Algorithm

For the Gibbs sampler we need $p(\theta_1|\theta_2, x_1, \ldots, x_n)$...but often we only have $g(\theta_1|\theta_2, x_1, \ldots, x_n) \propto p(\theta_1|\theta_2, x_1, \ldots, x_n)$

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How it works:

1. Pick an arbitrary point for the random walk.
2. Generate a candidate from a symmetric proposal distribution.
3. Compute $r = \frac{g(candidate)}{g(current)}$.
4. $$\text{Let new value} = \begin{cases} \text{candidate with probability min(r,1)} \\ \text{current, otherwise} \end{cases}$$
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Point: Likelihood and Prior are all we need!

Metropolis Algorithm Example

Earlier, we derived the posterior distribution of the proportion of females with lung cancer from a sample of 24 cancer subjects, 7 of which were female. In that example we used our previous knowledge of pdfs to make the integral in the denominator go to 1. Suppose we want to simply specify the prior and likelihood and employ the Metropolis Algorithm to take care of the rest.
Recall

- Prior: $p(\theta) = Beta(15, 15)$
- Likelihood: $p(x1, \ldots, x_n|\theta) = \theta^7(1-\theta)^{24-7}$
- Posterior:$p(\theta|x1, \ldots, x_n) = Beta(15+7, 24-7+15)$

See the "Metropolis Algorithm Beta 2021.R" file

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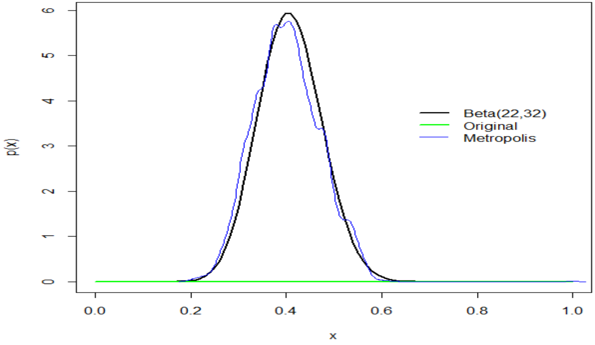

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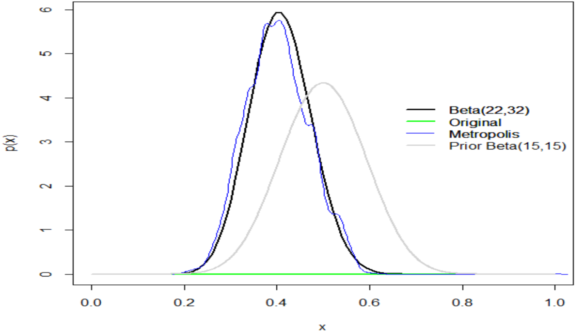

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Now suppose we can draw a random sample from $p(\theta|x_1, \ldots, x_n)$

sample 1 $(\theta_1^{(1)}, \ldots, \theta_k^{(1)})$
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- $E(\theta_2) \approx \frac{1}{B}\sum_{j=1}^{B} \theta_2^{(j)}$
- $E(g(\theta_1)) \approx \frac{1}{B}\sum_{j=1}^{B} g(\theta_1^{(j)})$